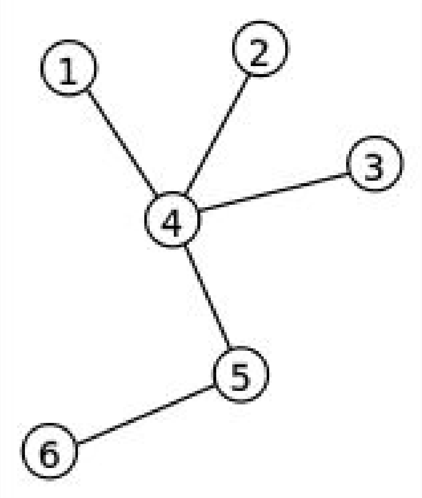
Minimum Spanning Trees

Problem Solving Club
January 25, 2017

Review: What is a tree?

A tree is an **undirected** graph. The following are all equivalent definitions:

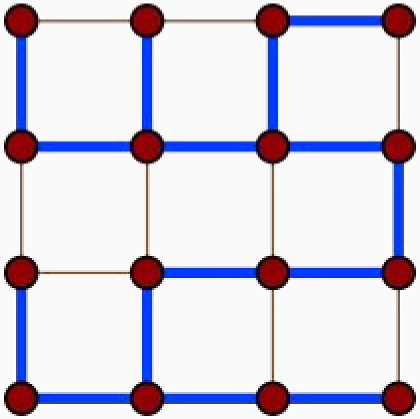
- Any two vertices are connected by exactly one path
- Connected with exactly V-1 edges
- Connected and has no cycles



What is a spanning tree?

A spanning tree of an **undirected** graph G is a tree that includes all vertices of G.

- Does every graph have a spanning tree?
- Can a graph have more than one spanning tree?
    - The number of spanning trees of any graph can be found using **Kirchhoff's theorem**
    - Take the **determinant** of a V×V matrix, where the entry in row i and column j is:
        - The degree of vertex i, if i = j
        - -1, if vertices i and j are adjacent
        - 0, otherwise

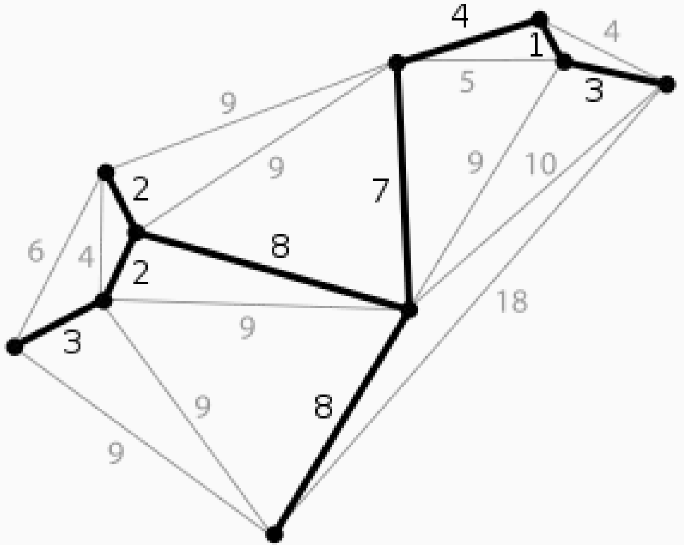


Minimum spanning trees

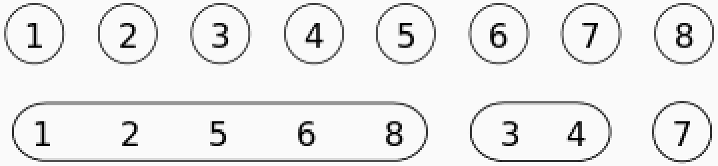
A **minimum spanning tree** is a spanning tree with the **minimum total edge weight**.

What are some practical applications for MST?

- The first MST algorithm was invented in 1926 to find an efficient electrical grid.
- Design of computer networks.
- Cluster analysis.



Disjoint-set (union-find) data structure



- Keeps track of a set of objects partitioned into disjoint subsets.
- Supports two operations:
  - **Find:** Determine which subset an object is in.
  - **Union:** Union two subsets.

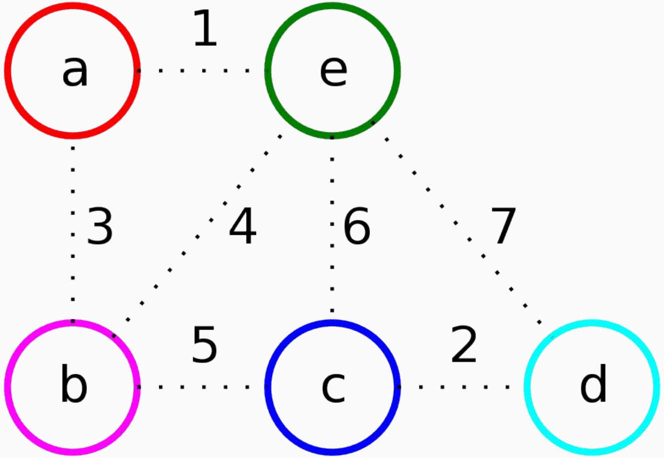
- It is possible to implement the operations in effectively constant time (inverse of Ackermann function).
- In programming contests, usually copy the (short) code from somewhere.

Kruskal’s algorithm

Kruskal’s algorithm is a **greedy** algorithm that finds a minimum spanning tree.

- Sort edges by ascending weight.
- While the tree is not complete:
    - Choose an edge with the lowest weight that has not been chosen yet.
    - Add the edge if it connects two different connected components.
- How to find a maximum spanning tree?

Edge	ab	ae	bc	be	cd	ed	ec
Weight	3	1	5	4	2	7	6



Example code for Kruskal's algorithm

```
bool edge_cmp(const edge &a, const edge &b) {
	return a.weight < b.weight;
}
vector<edge> mst(int n, vector<edge> edges) {
	union_find uf(n);
	sort(edges.begin(), edges.end(), edge_cmp);
	vector<edge> res;
	for (int i = 0; i < edges.size(); i++) {
		int u = edges[i].u, v = edges[i].v;
		if (uf.find(u) != uf.find(v)) {
			uf.unite(u, v);
			res.push_back(edges[i]);
		}
	}
	return res;
}
```

```
// Disjoint set data structure O(log n)

#define MAXN 1000
int p[MAXN];
int find(int x) {
	return p[x] == x ? x : p[x] = find(p[x]);
}
void unite(int x, int y) {
	p[find(x)] = find(y);
}

for (int i = 0; i < MAXN; i++) p[i] = i;
```